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The Oguchi approximation is exact for infinite-dimensional lattices

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Abstract. Spin systems are considered on sequences of lattices of increasing dimensionality. By dividing the lattices into suitable clusters with fixed internal interactions and scaling the external spin interactions inversely with the lattice dimension d, it is shown that the Oguchi effective field approximation becomes exact in the limit $d \to \infty$.

1. Introduction

For spin systems on hypercubic lattices, it is known (Pearce and Thompson 1978) that, if all the nearest-neighbour interactions are scaled inversely with the lattice dimension d, then the mean-field theory becomes exact in the limit $d \to \infty$. Mean-field theory, however, is only the first in a whole hierarchy of effective field approximations (Smart 1966) including the Oguchi, constant coupling and Bethe approximations.

In this paper we will be concerned with the Oguchi approximation (Oguchi 1955). For the spin-$\frac{1}{2}$ Ising model, Bowers (1981) has recently shown that the results of this approximation can be obtained by solving a model with repeated clusters of spins interacting via long-range interactions of equivalent neighbour type. Here we show that the Oguchi approximation can in fact be obtained exactly, without resorting to artificial long-range interactions, by solving a model with nearest-neighbour interactions on a d-dimensional lattice in the limit $d \to \infty$.

The rest of this section is devoted to a precise statement of the results for Ising-type (one-component) models. The methods are quite general, however, and, in the context of mean-field theory (i.e. single-site clusters), have previously been applied to n-vector (Thompson and Silver 1973), quantum Heisenberg (Pearce and Thompson 1975) and Potts models (Cant and Pearce 1983). The results on the Oguchi approximation are proved in §§ 2 and 3 by obtaining upper and lower bounds on the Ising model free energy that coalesce with the Oguchi free energy in the limit $d \to \infty$.

Suppose we are given a regular d-dimensional periodic lattice Λ with the sites divided into disjoint clusters $c_1, c_2, \ldots$, etc so that $\Lambda = \bigcup_l c_l$. Suppose further that, aside from location, these clusters are identical and are chosen and distributed in such a way that each lattice site is equivalent. A regular $(d+1)$-dimensional periodic lattice can then be formed by stacking these d-dimensional lattices, one immediately on top of the other, and identifying the top and bottom layers. In this way a sequence of higher and higher dimensional lattices is generated with d tending to infinity.

For each member of this sequence we consider the Hamiltonian H given by

$$H = -\sum_{\langle i,j\rangle \in B} J_{ij}\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i \tag{1.1}$$

where $\sigma_i \in [-1, 1]$ is an Ising spin assigned to each site i, and the sum over $\langle i, j\rangle$ extends over the set B of all nearest-neighbour pairs (i.e. bonds) of the lattice. In particular, we are interested in interactions of the form

$$J_{ij} = \begin{cases} K & \langle i,j\rangle \in C \\ J/2d & \langle i,j\rangle \in \tilde{C} \end{cases} \tag{1.2}$$

where C is the set of all bonds *within* clusters and its complement $\tilde{C} = B - C$ is the totality of bonds *between* clusters. We will always assume that J, K and the external magnetic field h are positive.

For a finite lattice of N clusters, each with n spins, the partition function is

$$Z_{nN} = \int_{-1}^{1} \cdots \int \prod_{i=1}^{nN} \mathrm{d}\nu(\sigma_i)\exp(-\beta H) \equiv \mathrm{Tr}\exp(-\beta H) \tag{1.3}$$

where $\beta = 1/k_{\mathrm{B}}T$ is the inverse temperature, $\mathrm{d}\nu(\sigma)$ is an (*a priori*) even probability measure on $[-1, 1]$ and Tr (trace) denotes the multiple integral. For the spin-$\frac{1}{2}$ Ising model,

$$\mathrm{d}\nu(\sigma) = \tfrac{1}{2}[\delta(\sigma+1) + \delta(\sigma-1)] \tag{1.4}$$

and the integrals in (1.3) reduce to the familiar sums. In the thermodynamic limit ($N \to \infty$), the free energy per spin ψ is given by

$$-\beta\psi = \lim_{N\to\infty} (nN)^{-1} \ln Z_{nN}. \tag{1.5}$$

It is now convenient to define the cluster Hamiltonian $H_c(h)$, for a prototypical cluster c, by

$$H_c(h) = -K\sum_{\langle i,j\rangle\in c}\sigma_i\sigma_j - h\sum_{i\in c}\sigma_i \tag{1.6}$$

where the sums are respectively over the bonds and sites in the cluster c. Given a Hamiltonian H, we also define expectations in the usual way by

$$\langle \ldots \rangle = \mathrm{Tr} \ldots \exp(-\beta H)/\mathrm{Tr}\exp(-\beta H). \tag{1.7}$$

The result we prove can now be stated as follows.

Theorem. Let ψ be the free energy (1.5) for the Ising model specified by (1.1)–(1.3). Suppose also that the cluster Hamiltonian (1.6) satisfies the generalised Hölder inequality

$$\left\langle \prod_{i\in c}\exp(x_i\sigma_i)\right\rangle_c \leq \prod_{i\in c}\left\langle \prod_{j\in c}\exp(x_i\sigma_j)\right\rangle_c^{1/n} \tag{1.8}$$

for any $x_1, x_2, \ldots, x_n \in \mathbb{R}$, where $\langle \ldots \rangle_c$ denotes the expectation (1.7) with respect to $H_c(h)$. Then

$$\lim_{d\to\infty}\psi = \psi_{\mathrm{Oguchi}} \tag{1.9}$$

where

$$\beta\psi_{\text{Oguchi}} = \min_m \left\{ \tfrac{1}{2}\beta J m^2 - n^{-1} \ln \operatorname{Tr} \exp\Big(\beta K \sum_{\langle i,j\rangle \in c} \sigma_i\sigma_j + \beta(Jm+h) \sum_{i\in c} \sigma_i \Big) \right\}. \tag{1.10}$$

Before proceeding to the proof of the theorem some remarks are in order.

(1) For $n=1$, (1.8) holds trivially and (1.10) reduces to the familiar mean-field result.

(2) For $n=2$, i.e. clusters consisting of pairs of spins, (1.8) holds by the Schwarz inequality. More generally, the generalised Hölder inequality (1.8) holds whenever the cluster interactions are reflection positive (Fröhlich and Lieb 1978, Fröhlich *et al* 1978). In particular, we can therefore take the basic cluster to be a triangle, square, cube etc with interactions along the edges. The proof of (1.8) is illustrated for the case of a triangle in the appendix.

(3) The number of spins in a cluster n can be allowed to tend to infinity with N. In this case (1.9) still holds provided the limit $n \to \infty$ is taken on the RHS of (1.10). By reflection positivity, we can therefore take the clusters to be a whole line of spins or a ladder of spins etc with nearest-neighbour interactions.

(4) An immediate corollary to the theorem is that, in the limit $d \to \infty$, the magnetisation is also given by the corresponding Oguchi expression. Specifically,

$$\lim_{d\to\infty} \langle\sigma_i\rangle = -\lim_{d\to\infty} \frac{\partial\psi}{\partial h} = -\frac{\partial}{\partial h}\psi_{\text{Oguchi}} = m_{\text{Oguchi}} \tag{1.11}$$

where the order of taking the limit and differentiating can be interchanged (Griffiths 1964) because the free energy ψ is a concave function of h. The Oguchi magnetisation m_{Oguchi} is the m that gives the minimum in (1.10), that is, it is the largest solution of the equation

$$m = \langle\sigma_i\rangle_m \tag{1.12}$$

where $\langle\ldots\rangle_m$ denotes the expectation (1.7) with respect to the cluster Hamiltonian $H_c(Jm+h)$.

2. Upper bound on the free energy

In this section we show that

$$\lim_{d\to\infty} \psi \leqslant \psi_{\text{Oguchi}}. \tag{2.1}$$

To begin we write the Hamiltonian (1.1) as

$$H = -K \sum_{\langle i,j\rangle\in C} \sigma_i\sigma_j - (J/2d) \sum_{\langle i,j\rangle\in \tilde{C}} \sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i = H_0 + H_1 \tag{2.2}$$

where

$$H_0 = -K \sum_{\langle i,j\rangle\in C} \sigma_i\sigma_j - (\tilde{J}m+h)\sum_{i\in\Lambda}\sigma_i + \tfrac{1}{2}nN\tilde{J}m^2 \tag{2.3}$$

$$H_1 = -(J/2d)\sum_{\langle i,j\rangle\in\tilde{C}} (m-\sigma_i)(m-\sigma_j). \tag{2.4}$$

Here m is arbitrary and

$$\tilde{J} = J|\tilde{C}|/nNd \tag{2.5}$$

where $|\tilde{C}|$ denotes the number of bonds in $\tilde{C} = B - C$, i.e. the number of bonds between clusters.

If we define the expectation $\langle \ldots \rangle_0$ with respect to H_0 as in (1.7), then Jensen's inequality tells us that

$$Z_{nN} = \langle \exp(-\beta H_1) \rangle_0 \operatorname{Tr} \exp(-\beta H_0) \geqslant \exp(-\beta \langle H_1 \rangle_0) \operatorname{Tr} \exp(-\beta H_0). \tag{2.6}$$

But

$$\langle H_1 \rangle_0 = -(J/2d) \sum_{\langle i,j \rangle \in \tilde{C}} (m - \langle \sigma_i \rangle_0)(m - \langle \sigma_j \rangle_0) \tag{2.7}$$

because

$$\langle \sigma_i \sigma_j \rangle_0 = \langle \sigma_i \rangle_0 \langle \sigma_j \rangle_0 \tag{2.8}$$

for i and j in different clusters. Choosing m to be a solution of the equation

$$m = \langle \sigma_i \rangle_0 \tag{2.9}$$

where the RHS depends on m but is independent of i, we find that

$$\langle H_1 \rangle_0 = 0. \tag{2.10}$$

Combining (2.6) with (2.10) and using (2.3), it follows that

$$-(nN)^{-1} \ln Z_{nN} \leqslant \tfrac{1}{2}\beta \tilde{J} m^2 - n^{-1} \ln \operatorname{Tr} \exp\left(\beta K \sum_{\langle i,j \rangle \in c} \sigma_i \sigma_j + \beta (\tilde{J} m + h) \sum_{i \in c} \sigma_i \right) \tag{2.11}$$

where we have factored the trace over the N clusters. In particular, since (2.9) is just the condition for the RHS of (2.11) to be stationary with respect to variations in m, we conclude that

$$-(nN)^{-1} \ln Z_{nN} \leqslant \min_m \left\{ \tfrac{1}{2}\beta \tilde{J} m^2 - n^{-1} \ln \operatorname{Tr} \exp\left(\beta K \sum_{\langle i,j \rangle \in c} \sigma_i \sigma_j + \beta (\tilde{J} m + h) \sum_{i \in c} \sigma_i \right) \right\}. \tag{2.12}$$

Recalling the definitions (1.5), (1.10) and (2.5), we now obtain the desired inequality (2.1) by taking the limits $N \to \infty$ and $d \to \infty$ and noticing that

$$\lim_{N,d \to \infty} \tilde{J} = J. \tag{2.13}$$

In this limit, the stationary equation (2.9) then becomes identical to the equation (1.12).

3. Lower bound on the free energy

In this section we show that

$$\lim_{d \to \infty} \psi \geqslant \psi_{\text{Oguchi}}. \tag{3.1}$$

This time we begin by writing the Hamiltonian (1.1) as

$$H = -(J/4d) \sum_{i,j \in \Lambda} A_{ij} \sigma_i \sigma_j - \tilde{K} \sum_{\langle i,j \rangle \in C} \sigma_i \sigma_j - h \sum_{i \in \Lambda} \sigma_i \tag{3.2}$$

where

$$\tilde{K} = K - J/2d \tag{3.3}$$

and

$$A_{ij} = \begin{cases} 1 & i \text{ and } j \text{ nearest neighbours} \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

is the adjacency matrix of the complete lattice.

To proceed with the derivation of the lower bound we wish to replace the matrix $(2d)^{-1}\boldsymbol{A}$ with a suitable positive definite matrix $\boldsymbol{K}$. Since $\boldsymbol{A}$ is a cyclic matrix it can be diagonalised by a unitary matrix $\boldsymbol{S}$, i.e.

$$\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \mathrm{diag}(\lambda_i) \tag{3.5}$$

where the λ_i are the eigenvalues of $\boldsymbol{A}$. We now define the matrix $\boldsymbol{K}$ by

$$K_{ij} = (2d)^{-1}|\boldsymbol{A}|_{ij} + \varepsilon\delta_{ij} \tag{3.6}$$

where $\varepsilon > 0$ and the non-negative definite matrix $|\boldsymbol{A}|$ is given by

$$|\boldsymbol{A}| = \boldsymbol{S}\ \mathrm{diag}(|\lambda_i|)\boldsymbol{S}^{-1}. \tag{3.7}$$

Since $|\boldsymbol{A}| - \boldsymbol{A}$ is a non-negative definite matrix, it follows immediately that

$$Z_{nN} \leqslant \mathrm{Tr}\exp\Big(\tfrac{1}{2}\beta \sum_{i,j\in\Lambda} K_{ij}\sigma_i\sigma_j + \beta\tilde{K}\sum_{\langle i,j\rangle\in C}\sigma_i\sigma_j + \beta h\sum_{i\in\Lambda}\sigma_i\Big). \tag{3.8}$$

We are now in a position to apply the standard indentity (Pearce and Thompson 1978) to the right side of (3.8). Doing this we obtain

$$\begin{aligned} Z_{nN} &\leqslant (\beta J/2\pi)^{nN/2}(\mathrm{Det}\,\boldsymbol{K})^{-1/2}\int_{\mathbb{R}^{nN}}\prod_{i\in\Lambda}\mathrm{d}x_i\exp\Big(-\tfrac{1}{2}\beta J\sum_{i,j\in\Lambda}K_{ij}^{-1}x_ix_j\Big) \\ &\quad\times\mathrm{Tr}\exp\Big(\beta\tilde{K}\sum_{\langle i,j\rangle\in C}\sigma_i\sigma_j + \beta\sum_{i\in\Lambda}(Jx_i+h)\sigma_i\Big) \\ &= (\beta J/2\pi)^{nN/2}(\mathrm{Det}\,\boldsymbol{K})^{-1/2}\int_{\mathbb{R}^{nN}}\prod_{i\in\Lambda}\mathrm{d}x_i\exp\Big(-\tfrac{1}{2}\beta J\sum_{i,j\in\Lambda}(K_{ij}^{-1} - z^{-1}\delta_{ij})x_ix_j\Big) \\ &\quad\times\prod_{l=1}^{N}\exp\Big(-\tfrac{1}{2}\beta Jz^{-1}\sum_{i\in c_l}x_i^2\Big)\mathrm{Tr}\exp\Big(\beta\tilde{K}\sum_{\langle i,j\rangle\in c_l}\sigma_i\sigma_j + \beta\sum_{i\in c_l}(Jx_i+h)\sigma_i\Big) \end{aligned} \tag{3.9}$$

where the last sums are over the bonds and sites in the cluster c_l. The generalised Hölder inequality (1.8), with the denominators cancelled on either side, can now be applied to the trace in (3.9) to obtain

$$\begin{aligned} Z_{nN} &\leqslant (\beta J/2\pi)^{nN/2}(\mathrm{Det}\,\boldsymbol{K})^{-1/2}\int_{\mathbb{R}^{nN}}\prod_{i\in\Lambda}\mathrm{d}x_i\exp\Big(-\tfrac{1}{2}\beta J\sum_{i,j\in\Lambda}(K_{ij}^{-1} - z^{-1}\delta_{ij})x_ix_j\Big) \\ &\quad\times\prod_{i\in\Lambda}\exp(-\tfrac{1}{2}\beta Jz^{-1}x_i^2)\Big[\mathrm{Tr}\exp\Big(\beta\tilde{K}\sum_{\langle i,j\rangle\in c_l}\sigma_i\sigma_j + \beta(Jx_i+h)\sum_{j\in c_l}\sigma_j\Big)\Big]^{1/n}. \end{aligned} \tag{3.10}$$

Maximising each term separately in the product occurring in (3.10), and performing the remaining Gaussian integrals, we find that

$$Z_{nN} \leqslant [\mathrm{Det}(\boldsymbol{I} - z^{-1}\boldsymbol{K})]^{-1/2}$$
$$\times \left\{ \max_m \exp\left[-\tfrac{1}{2}\beta J z^{-1} x^2 + n^{-1} \ln \mathrm{Tr} \exp\left(\beta \tilde{K} \sum_{\langle i,j\rangle \in c} \sigma_i \sigma_j + \beta (Jx + h) \sum_{i \in c} \sigma_i \right) \right] \right\}^{nN}. \tag{3.11}$$

This manipulation is valid only if the matrix $\boldsymbol{I} - z^{-1}\boldsymbol{K}$ is positive definite, that is, if $z > 1 + \varepsilon$. This condition can be relaxed to $z > 1$ if we let $\varepsilon \to 0+$. In this limit (3.11) still holds with the matrix $\boldsymbol{K}$ replaced with the matrix $(2d)^{-1}|\boldsymbol{A}|$.

Taking the thermodynamic limit $N \to \infty$, followed by the limits $d \to \infty$ and $z \to 1+$, in (3.11) using the definition (1.5) and the fact that

$$\lim_{d \to \infty} \tilde{K} = K \tag{3.12}$$

we finally obtain the desired inequality (3.1) provided

$$\lim_{z \to 1+} \lim_{d \to \infty} \lim_{N \to \infty} (nN)^{-1} \ln \mathrm{Det}(\boldsymbol{I} - |\boldsymbol{A}|/2dz) = 0. \tag{3.13}$$

This is proved in Pearce and Thompson (1978).

Appendix

In this appendix we prove the generalised Hölder inequality (1.8) for a triangle of spins $\sigma_1, \sigma_2, \sigma_3$ with pair interactions given by (1.6). Let us consider a reflection θ_1 given by

$$\theta_1 \sigma_1 = \sigma_1 \qquad \theta_1 \sigma_2 = \sigma_3 \qquad \theta_1 \sigma_3 = \sigma_2 \tag{A1}$$

and define θ_2 and θ_3 similarly. Given a real function $f = f(\sigma_1, \sigma_2)$, it is then clear that

$$\langle f \theta_1 f \rangle_c \geqslant 0 \tag{A2}$$

where $\theta_1 f = f(\theta_1 \sigma_1, \theta_1 \sigma_2) = f(\sigma_1, \sigma_3)$. This is reflection positivity. From it, the Schwarz inequality

$$\langle f \theta_1 g \rangle_c \leqslant \langle f \theta_1 f \rangle_c^{1/2} \langle g \theta_1 g \rangle_c^{1/2} \tag{A3}$$

where $g = g(\sigma_1, \sigma_2)$ follows by the usual proof.

Let us take

$$f = \exp(\tfrac{1}{2} x_1 \sigma_1 + x_2 \sigma_2) \qquad g = \exp(\tfrac{1}{2} x_1 \sigma_1 + x_3 \sigma_2). \tag{A4}$$

Then the Schwarz inequality (A3) becomes

$$\langle \exp(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3) \rangle_c \leqslant \langle \exp(x_1\sigma_1 + x_2\sigma_2 + x_2\sigma_3) \rangle_c^{1/2} \langle \exp(x_1\sigma_1 + x_3\sigma_2 + x_3\sigma_3) \rangle_c^{1/2}. \tag{A5}$$

By using similar inequalities for the other reflections, we thus obtain the symmetric inequality

$$\langle \exp(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3) \rangle_c \leqslant \prod_{\substack{i,j = 1,2,3 \\ i \neq j}} \langle \exp(x_i\sigma_1 + x_j\sigma_2 + x_j\sigma_3) \rangle_c^{1/6} \tag{A6}$$

where a product of six terms appears on the RHS. But now, setting $x_1 = x_2$ in (A5), we find

$$\langle \exp(x_1\sigma_1 + x_1\sigma_2 + x_3\sigma_3)\rangle_c \leqslant \langle \exp(x_1\sigma_1 + x_1\sigma_2 + x_1\sigma_3)\rangle_c^{1/2} \langle \exp(x_1\sigma_1 + x_3\sigma_2 + x_3\sigma_3)\rangle_c^{1/2}. \tag{A7}$$

Combining this with the inequality obtained by interchanging x_1 and x_3 now leads to the inequality

$$\langle \exp(x_1\sigma_1 + x_1\sigma_2 + x_3\sigma_3)\rangle_c \langle \exp(x_1\sigma_1 + x_3\sigma_2 + x_3\sigma_3)\rangle_c \leqslant \langle \exp(x_1\sigma_1 + x_1\sigma_2 + x_1\sigma_3)\rangle_c \langle \exp(x_3\sigma_1 + x_3\sigma_2 + x_3\sigma_3)\rangle_c. \tag{A8}$$

Putting (A8) in (A6) then finally yields the desired inequality

$$\langle \exp(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3)\rangle_c \leqslant \prod_{i=1}^{3} \langle \exp(x_i\sigma_1 + x_i\sigma_2 + x_i\sigma_3)\rangle_c^{1/3}. \tag{A9}$$

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